T_i-spaces, II

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Abstract

In this paper, we define the notion of fuzzy neighborhood filter at a set and we use it to introduce and study the fuzzy separation axioms T_3 and T_4. These axioms are defined using only the usual points and ordinary subsets as in the axioms $T_0,\ T_1,\ T_2$ which introduced and studied in part I. A similar study for $T_0,\ T_1,\ T_2$ will be done for these axioms.

Keywords: Fuzzy filters, Principal fuzzy filters, Fuzzy neighborhood filters, Valued fuzzy neighborhoods, Fuzzy topologies, Fuzzy separation axioms.

Introduction

In this second paper we continue to introduce and study the fuzzy separation axioms which are introduced some of them in the first part. We also continue the numbering of sections and begin therefore with Section 5. As in part I throughout this paper we use the same terminology.

Using the notion of fuzzy neighborhood filter at the points of a set we define, in this paper, the fuzzy neighborhood filter at this set. By means of the fuzzy neighborhood filter at a set and at a point the fuzzy separation axioms $T_3,\ T_4$ are defined. These axioms depends only on usual points and ordinary subsets so it is more general. Many properties for $T_3,\ T_4$ as in the cases ([12]) $T_0,\ T_1,\ T_2$ are fulfilled. For example: These fuzzy separation axioms are good extensions in sense of Lowen [14], that is, the induced fuzzy topological space $(X, \omega(T))$ is T_i if and only if the underlying topological space (X, T) is T_i for $i = 3, 4$. Moreover, each T_i-space is T_{i-1} for $i = 3, 4$. For each fuzzy topological space (X, τ) which is T_i, the α-level topological space (X, τ_α), $\alpha \in L_1$ and the initial topological space $(X, \iota(\tau))$ are T_i for $i = 3, 4$. Finally, the initial and final fuzzy topological spaces of a family of T_i-spaces, $i = 3, 4$, are also T_i-spaces and thus the fuzzy topological product space, subspace, sum space and quotient space of T_i-spaces, $i = 3, 4$, are also T_i. Our axioms are equivalent to the separation axioms defined by Gähler in [7] and [8].

5. T_3-Spaces

In this section we define the fuzzy neighborhood filter at a set and then, using this fuzzy neighborhood filter, notions of fuzzy regular spaces and T_3-spaces are introduced and studied.

For every fuzzy subset f of a non-empty set X, the fuzzy filter $[f]$ defined by [4]:

$$[f](g) = \bigvee_{f \wedge \overline{\alpha} \leq g} \sup(f \wedge \overline{\alpha}) \vee \bigvee_{\overline{\alpha} \leq g} \alpha$$

for all $g \in L^X$, is called *the superior principal fuzzy filter generated by f.* In case L is a complete chain and f is not constant, we have ([3]) for all $g \in L^X$:

$$[f](g) \ = \begin{cases} \sup f & if\ f \leq g, \\ \bigwedge\limits_{g(x) < f(x)} g(x) & otherwise. \end{cases}$$

For each subset M of X we have

$$[\chi_M] = \bigvee_{x \in M} \dot{x},$$

where χ_M is the characteristic function of M.

The fuzzy neighborhood filter $\mathcal{N}(x)$ at a point x is defined by Gähler in [6] and for the fuzzy neighborhood filter $\mathcal{N}(F)$ at a set $F \subseteq X$ we define it here by means of $\mathcal{N}(x), x \in F$ as:

$$\mathcal{N}(F) \;=\; \bigvee_{x\in F} \mathcal{N}(x).$$

It is clear that $\mathcal{N}(F)$ is a fuzzy filter on X and moreover,

$$\mathcal{N}(F) \geq [\chi_F].$$

Definition 5.1 A fuzzy topological space (X,τ) is called *regular* if $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist for all $x \in X, F \in P(X)$ with $F = \mathrm{cl}_\tau F$ and $x \notin F$.

Definition 5.2 A fuzzy topological space (X,τ) is called T_3 if it is regular and T_1.

The following Lemma is necessary to prove the next proposition.

Lemma 5.1 *For every fuzzy topological space (X,τ) and each $x \in X$ we have*

$$\mathrm{cl}\,\dot{x} = \dot{x} \textit{ implies } \mathrm{cl}_\tau\{x\} = \{x\}.$$

Proof. Let $\mathrm{cl}\,\dot{x} = \dot{x}$. Then $f(x) = \bigvee_{\mathrm{cl}_\tau g \leq f} g(x)$ for all $f \in L^X$ and since

$$\mathrm{cl}_\tau x_1(y) = \bigvee_{\mathcal{M}\leq\mathcal{N}(y)} \mathcal{M}(x_1) = \mathrm{int}_\tau x_1(y) = \bigvee_{\mathrm{cl}_\tau g\leq \mathrm{int}_\tau x_1} g(y) \leq \bigvee_{\mathrm{cl}_\tau g \leq x_1} g(y) = x_1(y).$$

Hence, $\mathrm{cl}_\tau x_1 = x_1$, that is, $\mathrm{cl}_\tau\{x\} = \{x\}$. $\square$

Proposition 5.1 *Every T_3-space is T_2-space.*

Proof. If (X,τ) is T_3 and $x \neq y$, then (X,τ) is T_1. By Theorem 3.1, we have $\mathrm{cl}\,\dot{x} = \dot{x}$ for all $x \in X$ and by means of Lemma 5.1, we have $\mathrm{cl}_\tau\{x\} = \{x\}$, since (X,τ) is regular, then $y \notin \{x\} = \mathrm{cl}_\tau\{x\}$ implies $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist. Hence, (X,τ) is T_2. $\square$

In the following theorem there will be introduced some equivalent definitions for the regular spaces.

Theorem 5.1 *For each fuzzy topological space (X,τ) the following are equivalent.*

(1) (X, τ) *is regular.*

(2) *For all* $x \in X$, $F \in P(X)$ *with* $F = \mathrm{cl}_\tau F$ *and* $x \notin F$ *we have* $\mathrm{cl}\,\mathcal{N}(x) \not\leq \mathcal{N}(y)$ *and* $\mathrm{cl}\,\mathcal{N}(y) \not\leq \mathcal{N}(x)$ *for each* $y \in F$.

(3) $\mathrm{cl}\,\mathcal{N}(x) = \mathcal{N}(x)$ *for each* $x \in X$.

(4) *For each* $x \in X$, *we have* $\mathcal{M} \leq \mathcal{N}(x)$ *implies* $\mathrm{cl}\,\mathcal{M} \leq \mathcal{N}(x)$ *for all fuzzy filters* $\mathcal{M}$ *on* X.

Proof. (1) $\Rightarrow$ (2): According to (1) we have $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist and then $\mathcal{N}(x) \wedge (\bigvee_{y \in F} \mathcal{N}(y)) = \bigvee_{y \in F} (\mathcal{N}(x) \wedge \mathcal{N}(y))$ does not exist. Hence, $\mathcal{N}(x) \wedge \mathcal{N}(y)$ for all $x \notin F$, $y \in F$, $x \neq y$ does not exist. From Proposition 4.1 we get $\dot{x} \not\leq \mathcal{N}(y)$ and $\dot{y} \not\leq \mathcal{N}(x)$ and therefore from Lemma 2.2. we get $\mathrm{cl}\,\mathcal{N}(x) \not\leq \mathcal{N}(y)$ and $\mathrm{cl}\,\mathcal{N}(y) \not\leq \mathcal{N}(x)$. Hence, (2) holds.

(2) $\Rightarrow$ (3): From (2) we have $\mathrm{cl}\,\mathcal{N}(x) \not\leq \mathcal{N}(y)$ for all $x \in X$, $F \in P(X)$ with $F = \mathrm{cl}_\tau F$ and $x \notin F$ and for each $y \in F$. Thus $\mathrm{cl}\,\mathcal{N}(x) \leq \mathcal{N}(z)$ for all $z \in X \setminus F$ and hence $\mathrm{cl}\,\mathcal{N}(x) \leq \mathcal{N}(x)$. That is, $\mathrm{cl}\,\mathcal{N}(x) \leq \mathcal{N}(x)$ for all $x \in X$. Therefore, $\mathcal{N}(x) = \mathrm{cl}\,\mathcal{N}(x)$ for all $x \in X$.

(3) $\Rightarrow$ (4): Let (3) be hold and $\mathcal{M} \leq \mathcal{N}(x)$ for each $x \in X$. Then from the property (1.6) of the closure operator we have $\mathrm{cl}\,\mathcal{M} \leq \mathrm{cl}\,\mathcal{N}(x)$. From $\mathrm{cl}\,\mathcal{N}(x) = \mathcal{N}(x)$ it follows $\mathrm{cl}\mathcal{M} \leq \mathcal{N}(x)$.

(4) $\Rightarrow$ (1): Let (4) be hold. Then $\mathrm{cl}\,\mathcal{N}(x) \leq \mathcal{N}(x)$ and hence $\mathcal{N}(x) = \mathrm{cl}\,\mathcal{N}(x)$ for all $x \in X$. Therefore, $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist for each $y \neq x$ and hence $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist for all $y \in F$ and $x \notin F$ with $F = \mathrm{cl}_\tau F$. Thus $\mathcal{N}(x) \wedge \mathcal{N}(F)$ fulfills the condition of regular space. That is, (1) holds. □

Condition (4) means if $\mathcal{M} \underset{\tau}{\rightarrow} x$, then also $\mathrm{cl}\,\mathcal{M} \underset{\tau}{\rightarrow} x$.

Example 5.1 For L is a complete chain and the space (X, τ) as in Example 4.2, where $X = \{x, y\}$ and $\tau = \{\overline{0}, \overline{1}, x_1, y_1\}$, let $x \in X$, $F = \{y\} = \mathrm{cl}_\tau F$ in $P(X)$, we get $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist, and also, for $y \in X$, $F = \{x\} = \mathrm{cl}_\tau F$ in $P(X)$, we get

$\mathcal{N}(F) \wedge \mathcal{N}(y)$ does not exist. Thus, (X, τ) is regular. We also can find $f = y_1$ and $g = x_1$ such that $f(x) < \mathcal{N}(y)(f)$ and $g(y) < \mathcal{N}(x)(g)$ and this means $\dot{x} \not\leq \mathcal{N}(y)$ and $\dot{y} \not\leq \mathcal{N}(x)$. Hence, (X, τ) is T_1 and thus (X, τ) is T_3.

Example 5.2 The indiscrete fuzzy topological space (X, τ), where $X = \{1, 2\}$, given in Example 2.1, is not T_1 and hence it is not T_3.

A topological space (X, T) is called *regular* if for all $x \in X$, $F \in P(X)$ with $F = \mathrm{cl}_\tau F$, $x \notin F$ there exist neighborhoods $\mathcal{O}_x$ of x and $\mathcal{O}_F$ of F such that $\mathcal{O}_x \cap \mathcal{O}_F = \emptyset$. (X, T) is called T_3 if it is regular and T_1.

Proposition 5.2 *A topological space (X, T) is T_3 if and only if the induced fuzzy topological space $(X, \omega(T))$ is T_3.*

Proof. By means of Proposition 3.2 we have (X, T) is T_1 equivalent to $(X, \omega(T))$ is T_1.

Now, let (X, T) be regular and let $x \notin F$ and $F = \mathrm{cl}_\tau F$. Then there are $\mathcal{O}_x \in T$ and $\mathcal{O}_F \in T$ such that $\mathcal{O}_x \cap \mathcal{O}_F = \emptyset$. If we take $f = \chi_{\mathcal{O}_x}$, $g = \chi_{\mathcal{O}_F}$, then from that $\chi_{\mathcal{O}_x}, \chi_{\mathcal{O}_F} \in \omega(T)$ hold we get

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(F)(g) = (\mathrm{int}_{\omega(T)} f)(x) \wedge \bigwedge_{y \in F} (\mathrm{int}_{\omega(T)} g)(y) = 1 > \sup(f \wedge g).$$

Hence, $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist.

Conversely, if $(X, \omega(T))$ is regular and $x \notin F = \mathrm{cl}_\tau F$, then there are $f, g \in L^X$ such that

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(F)(g) > \sup(f \wedge g).$$

This means

$$\mathrm{int}_{\omega(T)} f(x) \wedge \bigwedge_{y \in F} (\mathrm{int}_{\omega(T)} g(y)) > \sup(f \wedge g)$$

and hence

$$\mathrm{int}_{\omega(T)} f(x) > \sup(f \wedge g)$$

and

$$\text{int}_{\omega(T)}g(y) > \sup(f \wedge g) \text{ for each } y \in F.$$

Since $\text{int}_{\omega(T)}f, \text{int}_{\omega(T)}g \in \omega(T)$, taking $\alpha = \sup(f \wedge g)$, then $x \in s_\alpha(\text{int}_{\omega(T)}f) \in T$ and $y \in s_\alpha(\text{int}_{\omega(T)}g) \in T$ for each $y \in F$, that is, $s_\alpha(\text{int}_{\omega(T)}f) = \mathcal{O}_x$ and $s_\alpha(\text{int}_{\omega(T)}g) = \mathcal{O}_F$ are neighborhoods of x and F, respectively and moreover,

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g) > \alpha$$

for each $x \notin F$ implies $\mathcal{O}_x \cap \mathcal{O}_F = s_\alpha(\text{int}_{\omega(T)}f \wedge \text{int}_{\omega(T)}g) = \emptyset$. $\square$

In the following propositions will be shown that the initial fuzzy topological space (X, τ) of a family $((X_i, \tau_i))_{i \in I}$ of T_3-spaces is also T_3.

Remark 5.1 To show that the initial fuzzy topological space (X, τ) of a family $((X_i, \tau_i))_{i \in I}$ of T_0, T_1, T_2-spaces the mappings f_i for some $i \in I$ must be injective but in the case of T_3, T_4 will be shown that the mappings f_i must be also closed.

At first consider the case of I being a singleton.

Proposition 5.3 *Let (Y, σ) be a T_3-space and let $f : X \to Y$ be an injective fuzzy closed mapping. Then the initial fuzzy topological space $(X, f^{-1}(\sigma))$ is also T_3.*

Proof. From Proposition 3.4 it follows that $(X, f^{-1}(\sigma))$ is T_1-space.

Now let $x \in X$ and F a closed subset of X with $x \notin F$. Since f is injective and closed, then $f(x) \notin f(F)$ and $f(F)$ is a closed subset of Y and from that (Y, σ) is regular it follows $\mathcal{N}(f(x)) \wedge \mathcal{N}(f(F))$ does not exist, that is, there exist $g, h \in L^Y$ such that

$$\mathcal{N}(f(x))(g) \wedge \mathcal{N}(f(F))(h) > \sup(g \wedge h)$$

and this means

$$(\text{int}_\sigma g)(f(x)) \wedge \bigwedge_{y \in f(F)} (\text{int}_\sigma h)(y) = (\text{int}_\sigma g)(f(x)) \wedge \bigwedge_{z \in F} (\text{int}_\sigma h)(f(z)) > \sup(g \wedge h).$$

Because of that $f : (X, f^{-1}(\sigma)) \to (Y, \sigma)$ is fuzzy continuous we have $(\text{int}_\sigma g) \circ f \leq \text{int}_{f^{-1}(\sigma)}(g \circ f)$ for all $g \in L^Y$ and we have also $\sup(g \wedge h) \geq \sup((g \circ f) \wedge (h \circ f))$ hence we get

$$(\text{int}_{f^{-1}(\sigma)}(g \circ f))(x) \wedge \bigwedge_{z \in F} (\text{int}_{f^{-1}(\sigma)}(h \circ f))(z) > \sup((g \circ f) \wedge (h \circ f)).$$

Thus there exist $k = g \circ f, l = h \circ f \in L^X$ such that

$$(\text{int}_{f^{-1}(\sigma)} k)(x) \wedge \bigwedge_{z \in F} (\text{int}_{f^{-1}(\sigma)} l)(z) > \sup(k \wedge l).$$

Hence, $(X, f^{-1}(\sigma))$ is a regular space. This means $(X, f^{-1}(\sigma))$ is T_1 and regular and therefore it is T_3-space. □

Now consider the case of I be an arbitrary class.

Proposition 5.4 *Let (X_i, τ_i) be a T_3-space for all $i \in I$ and let $f_i : X \to X_i$, for some $i \in I$, be an injective fuzzy closed mapping. Then the initial fuzzy topological space (X, τ) is also T_3.*

Proof. Proposition 3.5 shows that (X, τ) is T_1-space.

If $x \in X$ and F is a closed subset of X with $x \notin F$, then f_i is injective and closed imply $f_i(x) \notin f_i(F)$ and $f_i(F)$ is a closed subset of X_i and from that (X_i, τ_i) is regular it follows $\mathcal{N}(f_i(x)) \wedge \mathcal{N}(f_i(F))$ does not exist, that is, there exist $\lambda_i,\ \mu_i \in L^{X_i}$ such that

$$(\text{int}_{\tau_i}\lambda_i)(f_i(x)) \wedge \bigwedge_{y \in f_i(F)} (\text{int}_{\tau_i}\mu_i)(y) = (\text{int}_{\tau_i}\lambda_i)(f_i(x)) \wedge \bigwedge_{z \in F} (\text{int}_{\tau_i}\mu_i)(f_i(z)) > \sup(\lambda_i \wedge \mu_i).$$

Since f_i is fuzzy continuous, then $(\text{int}_{\tau_i}\lambda_i) \circ f_i \leq \text{int}_\tau(\lambda_i \circ f_i)$ for all $\lambda_i \in L^{X_i}$. Hence

$$\text{int}_\tau(\lambda_i \circ f_i)(x) \wedge \bigwedge_{z \in F} (\text{int}_\tau(\mu_i \circ f_i))(z) > \sup(\lambda_i \wedge \mu_i) \geq \sup((\lambda_i \circ f_i) \wedge (\mu_i \circ f_i)).$$

This means we have $\lambda = \lambda_i \circ f_i \in L^X,\ \mu = \mu_i \circ f_i \in L^X$ such that

$$(\text{int}_\tau \lambda)(x) \wedge \bigwedge_{z \in F} (\text{int}_\tau \mu)(z) > \sup(\lambda \wedge \mu).$$

Hence, (X,τ) is a regular space and therefore it is T_3-space. □

The following result is a direct consequence of Propositions 5.3 and 5.4.

Corollary 5.1 The fuzzy topological subspace and the fuzzy topological product space of T_3-spaces are also T_3.

Now we shall show that the final fuzzy topological space (X,τ) of a family $((X_i,\tau_i))_{i\in I}$ of T_3-spaces is also T_3.

Proposition 5.5 *If (X,τ) is a T_3-space and $f : X \to Y$ a surjective fuzzy open mapping, then the final fuzzy topological space $(Y, f(\tau))$ is also T_3.*

Proof. Let $y \in Y$ and F be a closed subset of Y with $y \notin F$. Since f is surjective and continuous, then $f^{-1}(y) \in X$ and $f^{-1}(F)$ is a closed subset of X with $f^{-1}(y) \notin f^{-1}(F)$. From that (X,τ) is a regular space it follows $\mathcal{N}(f^{-1}(y)) \wedge \mathcal{N}(f^{-1}(F))$ does not exist, that is, there are $g, h \in L^X$ such that

$$(\mathrm{int}_\tau g)(f^{-1}(y)) \wedge \bigwedge_{z\in f^{-1}(F)} (\mathrm{int}_\tau h)(z) > \sup(g \wedge h)$$

which means

$$(\mathrm{int}_\tau g)(f^{-1}(y)) \wedge \bigwedge_{x\in F} (\mathrm{int}_\tau h)(f^{-1}(x)) > \sup(g \wedge h)$$

and this means

$$(f(\mathrm{int}_\tau g))(y) \wedge \bigwedge_{x\in F} (f(\mathrm{int}_\tau h))(x) > \sup(g \wedge h).$$

Since f is fuzzy open, it follows $f(\mathrm{int}_\tau g) \le \mathrm{int}_{f(\tau)}(f(g))$ for all $g \in L^X$ and therefore

$$(\mathrm{int}_{f(\tau)} f(g))(y) \wedge \bigwedge_{x\in F} (\mathrm{int}_{f(\tau)} f(h))(x) > \sup(g \wedge h) \ge \sup(f(g) \wedge f(h)).$$

Since $f(g), f(h) \in L^Y$, then we get that the final fuzzy topological space $(Y, f(\tau))$ is regular. From Proposition 3.6 we have $(Y, f(\tau))$ is T_1-space and therefore it is T_3-space. □

Proposition 5.6 *Let I be any class and (X_i, τ_i) be a T_3-space for all $i \in I$ and $f_i : X_i \to X$ be a surjective fuzzy open mapping for some $i \in I$. Then the final fuzzy topological space (X, τ) is also T_3.*

Proof. Let $x \in X$ and F be a closed subset of X with $x \notin F$. Since f_i is surjective and continuous, then $f_i^{-1}(x) \in X_i$ and $f_i^{-1}(F)$ is a closed subset of X_i with $f_i^{-1}(x) \notin f_i^{-1}(F)$. From that (X_i, τ_i) is regular it follows there are $\lambda_i,\ \mu_i \in L^{X_i}$ such that

$$(\mathrm{int}_{\tau_i}\lambda_i)(f_i^{-1}(x)) \wedge \bigwedge_{y \in f_i^{-1}(F)} (\mathrm{int}_{\tau_i}\mu_i)(y) > \sup(\lambda_i \wedge \mu_i)$$

which means

$$(\mathrm{int}_{\tau_i}\lambda_i)(f_i^{-1}(x)) \wedge \bigwedge_{z \in F} (\mathrm{int}_{\tau_i}\mu_i)(f_i^{-1}(z)) > \sup(\lambda_i \wedge \mu_i)$$

and this means

$$(f_i(\mathrm{int}_{\tau_i}\lambda_i))(x) \wedge \bigwedge_{z \in F} (f_i(\mathrm{int}_{\tau_i}\mu_i))(z) > \sup(\lambda_i \wedge \mu_i).$$

Since f_i is fuzzy open, it follows $f_i(\mathrm{int}_{\tau_i}\lambda_i) \leq \mathrm{int}_\tau(f_i(\lambda_i))$ for all $\lambda_i \in L^{X_i}$ and therefore

$$\mathrm{int}_\tau f_i(\lambda_i)(x) \wedge \bigwedge_{z \in F} \mathrm{int}_\tau f_i(\mu_i)(z) > \sup(\lambda_i \wedge \mu_i) \ = \ \sup(f_i(\lambda_i) \wedge f_i(\mu_i)).$$

Since $f_i(\lambda_i), f_i(\mu_i) \in L^X$, then we get that the final fuzzy topological space (X, τ) is regular. Proposition 3.7 states that (X, τ) is T_1-space and therefore it is T_3-space. □

The following result is a direct consequence of Propositions 5.5 and 5.6.

Corollary 5.2 The fuzzy topological sum space and the fuzzy topological quotient space of T_3-spaces are also T_3.

In the following it will be shown that the finer fuzzy topological space of T_3 is also T_3.

Proposition 5.7 *Let (X, τ) be a T_3-space and let σ be a fuzzy topology on X finer than τ. Then (X, σ) is also T_3-space.*

Proof. Let $x \in X$ and F be a closed subset of X with $x \notin F$ and let $\mathcal{N}_\tau(x)$, $\mathcal{N}_\sigma(x)$ be the fuzzy neighborhood filters of the spaces (X, τ), (X, σ), respectively, at x. Since (X, τ) is regular, then $\mathcal{N}_\tau(x) \wedge \mathcal{N}_\tau(F)$ does not exist. σ is finer than τ implies $\mathcal{N}_\sigma(x) \leq \mathcal{N}_\tau(x)$ and $\mathcal{N}_\sigma(F) \leq \mathcal{N}_\tau(F)$ and thus $\mathcal{N}_\sigma(x) \wedge \mathcal{N}_\sigma(F) \leq \mathcal{N}_\tau(x) \wedge \mathcal{N}_\tau(F)$. Hence $\mathcal{N}_\sigma(x) \wedge \mathcal{N}_\sigma(F)$ does not exist and therefore (X, σ) is also regular. Proposition 3.8 states that (X, σ) is T_1-space and thus it is T_3-space. □

6. T_4-Spaces

Using the neighborhood filter at a set, defined in the last section, we define here a notion of the fuzzy normal space and T_4-space.

Definition 6.1 A fuzzy topological space (X, τ) is called *normal* if for all $F_1, F_2 \in P(X)$ with $F_1 = \mathrm{cl}_\tau F_1, F_2 = \mathrm{cl}_\tau F_2$ and $F_1 \cap F_2 = \emptyset$ we have $\mathcal{N}(F_1) \wedge \mathcal{N}(F_2)$ does not exist.

Definition 6.2 A fuzzy topological space (X, τ) is called T_4 if it is normal and T_1.

Proposition 6.1 *Every T_4-space is T_3-space.*

Proof. If (X, τ) is T_4, then it is T_1 and thus by Lemma 5.1 $\mathrm{cl}_\tau\{x\} = \{x\}$ for all $x \in X$. Thus $x \notin F = \mathrm{cl}_\tau F$ implies we have $F_1 = \{x\} = \mathrm{cl}_\tau\{x\}$ and $F_2 = F = \mathrm{cl}_\tau F$ with $F_1 \cap F_2 = \emptyset$ and hence $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist. That is, (X, τ) is regular and it is T_1. Therefore, (X, τ) is T_3. □

Theorem 6.1 *Let (X, τ) be a fuzzy topological space. Then the following are equivalent.*

(1) *(X, τ) is normal.*

(2) *For all* $F_1, F_2 \in P(X)$ *with* $F_1 = \mathrm{cl}_\tau F_1$, $F_2 = \mathrm{cl}_\tau F_2$ *and* $F_1 \cap F_2 = \emptyset$ *we have* $\mathrm{cl}\,\mathcal{N}(F_1) \not\leq \mathcal{N}(F_2)$ *and* $\mathrm{cl}\,\mathcal{N}(F_2) \not\leq \mathcal{N}(F_1)$.

(3) $\mathrm{cl}\,\mathcal{N}(F) = \mathcal{N}(F)$ *for all* $F \in P(X)$ *with* $F = \mathrm{cl}F$.

(4) *For all* $F \in P(X)$ *with* $F = \mathrm{cl}_\tau F$ *we have* $\mathcal{M} \leq \mathcal{N}(F)$ *implies* $\mathrm{cl}\,\mathcal{M} \leq \mathcal{N}(F)$ *for all fuzzy filters* $\mathcal{M}$ *on* X.

Proof. (1) $\Rightarrow$ (2): For all $F_1, F_2 \in P(X)$ with $F_i = \mathrm{cl}_\tau F_i$, $i = 1, 2$ and $F_1 \cap F_2 = \emptyset$ we have $\mathcal{N}(F_1) \wedge \mathcal{N}(F_2)$ does not exist and hence $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist for all $x \in F_1$ and $y \in F_2$. Thus by means of Lemma 2.2, $\mathrm{cl}\,\mathcal{N}(x) \not\leq \mathcal{N}(y)$ and $\mathrm{cl}\,\mathcal{N}(y) \not\leq \mathcal{N}(x)$ and therefore $\mathrm{cl}\,\mathcal{N}(F_1) \not\leq \mathcal{N}(F_2)$ and $\mathrm{cl}\,\mathcal{N}(F_2) \not\leq \mathcal{N}(F_1)$.

(2) $\Rightarrow$ (3): Let (2) be hold. Then $\mathrm{cl}\,\mathcal{N}(F) \not\leq \mathcal{N}(G)$ for all $F, G \in P(X)$ with $F = \mathrm{cl}_\tau F$, $G = \mathrm{cl}_\tau G$ and $G \subseteq X \setminus F$. Hence, $\mathrm{cl}\,\mathcal{N}(F) \leq \mathcal{N}(H)$ for all $H \subseteq F$ and thus $\mathrm{cl}\,\mathcal{N}(F) \leq \mathcal{N}(F)$. Hence, $\mathcal{N}(F) = \mathrm{cl}\,\mathcal{N}(F)$ for all $F \in P(X)$ with $F = \mathrm{cl}_\tau F$.

(3) $\Rightarrow$ (4): Let (3) be hold. Then $\mathcal{N}(F) = \mathrm{cl}\,\mathcal{N}(F)$ holds, $\mathcal{M} \leq \mathcal{N}(F)$ implies $\mathrm{cl}\,\mathcal{M} \leq \mathrm{cl}\,\mathcal{N}(F) = \mathcal{N}(F)$. Hence, (4) holds.

(4) $\Rightarrow$ (1): Let $F_1, F_2 \in P(X)$ with $F_i = \mathrm{cl}_\tau F_i$, $i = 1, 2$ and $F_1 \cap F_2 = \emptyset$. (4) implies $\mathrm{cl}\,\mathcal{N}(F_i) = \mathcal{N}(F_i)$, $i = 1, 2$ and hence $\mathcal{N}(F_1) \wedge \mathcal{N}(F_2)$ does not exist. $\square$

Example 6.1 Since in the space (X, τ) in Example 5.1, where $X = \{x, y\}$, $\tau = \{\overline{0}, \overline{1}, x_1, y_1\}$, we have $\{x\}$ and $\{y\}$ are the only closed sets which fulfill the condition of the normal space and $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist. Hence, (X, τ) is normal and since (X, τ) is T_1 it follows (X, τ) is T_4.

A topological space (X, T) is called *normal* if for all $F_1 \in P(X)$,$F_2 \in P(X)$ with $F_1 = \mathrm{cl}_\tau F_1$, $F_2 = \mathrm{cl}_\tau F_2$ there exist neighborhoods $\mathcal{O}_{F_1}$ and $\mathcal{O}_{F_2}$ such that $\mathcal{O}_{F_1} \cap \mathcal{O}_{F_2} = \emptyset$. (X, T) is called T_4 if it is normal and T_1.

Proposition 6.2 *A topological space* (X, T) *is* T_4 *if and only if the induced fuzzy topological space* $(X, \omega(T))$ *is* T_4.

Proof. From Proposition 3.2 we get (X,T) is T_1 if and only if $(X,\omega(T))$ is T_1. If (X,T) is normal and $F_1,F_2\in P(X)$, $F_i=\mathrm{cl}_\tau F_i$, $i=1,2$ and $F_1\cap F_2=\emptyset$, then there are $\mathcal{O}_{F_1},\mathcal{O}_{F_2}\in T$ such that $\mathcal{O}_{F_1}\cap\mathcal{O}_{F_2}=\emptyset$. Hence, there are $f=\chi_{\mathcal{O}_{F_1}}, g=\chi_{\mathcal{O}_{F_2}}\in L^X$ for which

$$\mathcal{N}(F_1)(f)\wedge\mathcal{N}(F_2)(g)=\bigwedge_{x\in F_1}\mathrm{int}_{\omega(T)}f(x)\wedge\bigwedge_{y\in F_2}\mathrm{int}_{\omega(T)}g(y)=1>\sup(f\wedge g).$$

Thus $\mathcal{N}(F_1)\wedge\mathcal{N}(F_2)$ does not exist, that is, $(X,\omega(T))$ is normal.

Conversely, let $(X,\omega(T))$ be normal and let $F_i=\mathrm{cl}_\tau F_i$ for $i=1,2$ and $F_1\cap F_2=\emptyset$. Then there are $f,g\in L^X$ for which

$$\bigwedge_{x\in F_1}(\mathrm{int}_{\omega(T)}f(x))\wedge\bigwedge_{y\in F_2}(\mathrm{int}_{\omega(T)}g(y))>\sup(f\wedge g). \tag{6.1}$$

Since, $\mathrm{int}_{\omega(T)}f\in\omega(T)$ and $\mathrm{int}_{\omega(T)}f(x)>\sup(f\wedge g)$ for each $x\in F_1$, then, taking $\alpha=\sup(f\wedge g)$, we get $F_1\subseteq s_\alpha(\mathrm{int}_{\omega(T)}f)$ and $s_\alpha(\mathrm{int}_{\omega(T)}f)\in T$. Similarly, we get $F_2\subseteq s_\alpha(\mathrm{int}_{\omega(T)}g)\in T$. Hence, there are neighborhoods $\mathcal{O}_{F_1}=s_\alpha(\mathrm{int}_{\omega(T)}f)$ and $\mathcal{O}_{F_2}=s_\alpha(\mathrm{int}_{\omega(T)}g)$ of F_1 and F_2, respectively, and moreover because of (6.1) we get

$$\mathcal{O}_{F_1}\cap\mathcal{O}_{F_2}=s_\alpha(\mathrm{int}_{\omega(T)}f\wedge\mathrm{int}_{\omega(T)}g)=\emptyset.$$

Thus (X,T) is normal. □

The following propositions show that the initial fuzzy topological space (X,τ) of a family $((X_i,\tau_i))_{i\in I}$ of T_4-spaces, whenever $f_i:X\to X_i$ for some $i\in I$ is an injective fuzzy closed mapping, is also T_4.

For the case of I being a singleton we get the following result.

Proposition 6.3 *Let (Y,σ) be a T_4-space and let $f:X\to Y$ be an injective fuzzy closed mapping. Then the initial fuzzy topological space $(X,f^{-1}(\sigma))$ is also T_4.*

Proof. Let F, G be disjoint closed subsets of X. From that f is injective and closed it follows $f(F)$, $f(G)$ are also disjoint closed subsets of Y. Since (Y,σ) is normal space, then $\mathcal{N}(f(F)) \wedge \mathcal{N}(f(G))$ does not exist, that is, there exist $g, h \in L^Y$ such that

$$\bigwedge_{x\in f(F)} (\mathrm{int}_\sigma g)(x) \wedge \bigwedge_{y\in f(G)} (\mathrm{int}_\sigma h)(y) > \sup(g \wedge h)$$

and this means

$$\bigwedge_{z\in F} (\mathrm{int}_\sigma g)(f(z)) \wedge \bigwedge_{w\in G} (\mathrm{int}_\sigma h)(f(w)) > \sup(g \wedge h)$$

which means

$$\bigwedge_{z\in F} ((\mathrm{int}_\sigma g) \circ f)(z) \wedge \bigwedge_{w\in G} ((\mathrm{int}_\sigma h) \circ f)(w) > \sup((g \circ f) \wedge (h \circ f)).$$

Because of that $f : (X, f^{-1}(\sigma)) \to (Y, \sigma)$ is fuzzy continuous it follows $(\mathrm{int}_\sigma g) \circ f \leq \mathrm{int}_{f^{-1}(\sigma)}(g \circ f)$ for all $g \in L^Y$ and thus we have

$$\bigwedge_{z\in F} (\mathrm{int}_{f^{-1}(\sigma)}(g \circ f))(z) \wedge \bigwedge_{w\in G} (\mathrm{int}_{f^{-1}(\sigma)}(h \circ f))(w) > \sup((g \circ f) \wedge (h \circ f)).$$

Thus there exist $k = g \circ f, l = h \circ f \in L^X$ such that

$$\bigwedge_{z\in F} (\mathrm{int}_{f^{-1}(\sigma)} k)(z) \wedge \bigwedge_{w\in G} (\mathrm{int}_{f^{-1}(\sigma)} l)(w) > \sup(k \wedge l).$$

Hence, $(X, f^{-1}(\sigma))$ is a normal space. From Proposition 3.4 it follows that the space $(X, f^{-1}(\sigma))$ is T_1 and therefore it is T_4-space. □

Now consider the case of I be any class.

Proposition 6.4 *Let (X_i, τ_i) be a T_4-space for all $i \in I$ and let $f_i : X \to X_i$, for some $i \in I$, be an injective fuzzy closed mapping. Then the initial fuzzy topological space (X, τ) is also T_4.*

Proof. If F, G are disjoint closed subsets of X, then f_i is injective and closed imply $f_i(F)$, $f_i(G)$ are also disjoint closed subsets of X_i. From that (X_i, τ_i) is normal space it follows $\mathcal{N}(f_i(F)) \wedge \mathcal{N}(f_i(G))$ does not exist, that is, there exist λ_i, $\mu_i \in L^{X_i}$ such that

$$\bigwedge_{x\in f_i(F)} (\mathrm{int}_{\tau_i} \lambda_i)(x) \wedge \bigwedge_{y\in f_i(G)} (\mathrm{int}_{\tau_i} \mu_i)(y) > \sup(\lambda_i \wedge \mu_i).$$

Therefore

$$\bigwedge_{z\in F}(\mathrm{int}_{\tau_i}\lambda_i)(f_i(z)) \wedge \bigwedge_{w\in G}(\mathrm{int}_{\tau_i}\mu_i)(f_i(w)) > \sup(\lambda_i \wedge \mu_i)$$

and this means

$$\bigwedge_{z\in F}((\mathrm{int}_{\tau_i}\lambda_i)\circ f_i)(z) \wedge \bigwedge_{w\in G}((\mathrm{int}_{\tau_i}\mu_i)\circ f_i)(w) > \sup((\lambda_i\circ f_i)\wedge(\mu_i\circ f_i)).$$

Since f_i is fuzzy continuous, then

$$\bigwedge_{z\in F}(\mathrm{int}_{\tau}(\lambda_i\circ f_i))(z) \wedge \bigwedge_{w\in G}(\mathrm{int}_{\tau}(\mu_i\circ f_i))(w) > \sup((\lambda_i\circ f_i)\wedge(\mu_i\circ f_i)).$$

This means we have $\lambda=\lambda_i\circ f_i \in L^X,\ \mu=\mu_i\circ f_i\in L^X$ such that

$$\bigwedge_{z\in F}(\mathrm{int}_{\tau}\lambda)(z) \wedge \bigwedge_{w\in G}(\mathrm{int}_{\tau}\mu)(w) > \sup(\lambda\wedge\mu).$$

Hence, (X,τ) is a normal space. From Proposition 3.5 we have (X,τ) is T_1-space and therefore it is T_4-space. □

The following result is a direct consequence of Propositions 6.3 and 6.4.

Corollary 6.1 The fuzzy topological subspace and the fuzzy topological product space of T_4-spaces are also T_4.

Now we are going to show that the final fuzzy topological space (X,τ) of a family $((X_i,\tau_i))_{i\in I}$ of T_4-spaces is also T_4.

Proposition 6.5 *If (X,τ) is a T_4-space and $f: X\to Y$ a surjective fuzzy open mapping, then the final fuzzy topological space $(Y, f(\tau))$ is also T_4.*

Proof. Let $F,\ G$ be disjoint closed subsets of Y. Since f is surjective and continuous, then $f^{-1}(F),\ f^{-1}(G)$ are also disjoint closed subsets of X. From that (X,τ) is normal space it follows $\mathcal{N}(f^{-1}(F))\wedge\mathcal{N}(f^{-1}(G))$ does not exist, that is, there are $g,h\in L^X$ such that

$$\bigwedge_{z\in f^{-1}(F)}(\mathrm{int}_{\tau}g)(z) \wedge \bigwedge_{w\in f^{-1}(G)}(\mathrm{int}_{\tau}h)(w) > \sup(g\wedge h)$$

which means

$$\bigwedge_{x\in F}(\mathrm{int}_\tau g)(f^{-1}(x))\wedge\bigwedge_{y\in G}(\mathrm{int}_\tau h)(f^{-1}(y))>\sup(g\wedge h)$$

and this means

$$\bigwedge_{x\in F}(f(\mathrm{int}_\tau g))(x)\wedge\bigwedge_{y\in G}(f(\mathrm{int}_\tau h))(y)>\sup(g\wedge h).$$

Since f is fuzzy open, it follows $f(\mathrm{int}_\tau g)\leq \mathrm{int}_{f(\tau)}(f(g))$ for all $g\in L^X$ and therefore

$$\bigwedge_{x\in F}(\mathrm{int}_{f(\tau)}f(g))(x)\wedge\bigwedge_{y\in G}(\mathrm{int}_{f(\tau)}f(h))(y)>\sup(f(g)\wedge f(h)).$$

Since $f(g), f(h)\in L^Y$, then we get that the final fuzzy topological space $(Y, f(\tau))$ is normal. From Proposition 3.6 we have $(Y, f(\tau))$ is T_1-space and therefore it is T_4-space. □

Proposition 6.6 *Let I be any class and (X_i,τ_i) be a T_4-space for all $i\in I$ and $f_i : X_i\to X$ be a surjective fuzzy open mapping for some $i\in I$. Then the final fuzzy topological space (X,τ) is also T_4.*

Proof. Let F, G be disjoint closed subsets of X. Since f_i is surjective and continuous, then $f_i^{-1}(F)$, $f_i^{-1}(G)$ are also disjoint closed subsets of X_i. Because of that (X_i,τ_i) is normal it follows there are $\lambda_i,\ \mu_i\in L^{X_i}$ such that

$$\bigwedge_{z\in f_i^{-1}(F)}(\mathrm{int}_{\tau_i}\lambda_i)(z)\wedge\bigwedge_{w\in f_i^{-1}(G)}(\mathrm{int}_{\tau_i}\mu_i)(w)>\sup(\lambda_i\wedge\mu_i)$$

which means

$$\bigwedge_{x\in F}(\mathrm{int}_{\tau_i}\lambda_i)(f_i^{-1}(x))\wedge\bigwedge_{y\in G}(\mathrm{int}_{\tau_i}\mu_i)(f_i^{-1}(y))>\sup(\lambda_i\wedge\mu_i)$$

and this means

$$\bigwedge_{x\in F}(f_i(\mathrm{int}_{\tau_i}\lambda_i))(x)\wedge\bigwedge_{y\in G}(f_i(\mathrm{int}_{\tau_i}\mu_i))(y)>\sup(\lambda_i\wedge\mu_i).$$

Since f_i is fuzzy open, it follows $f_i(\text{int}_{\tau_i}\lambda_i) \leq \text{int}_\tau(f_i(\lambda_i))$ for all $\lambda_i \in L^{X_i}$ and therefore

$$\bigwedge_{x\in F}(\text{int}_\tau f_i(\lambda_i))(x) \wedge \bigwedge_{y\in G}(\text{int}_\tau f_i(\mu_i))(y) > \sup(f_i(\lambda_i) \wedge f_i(\mu_i)).$$

Since $f_i(\lambda_i), f_i(\mu_i) \in L^X$, then we get that the final fuzzy topological space (X, τ) is normal. From Proposition 3.7 it follows that (X, τ) is T_1-space and hence it is T_4-space. □

The following result is a direct consequence of Propositions 6.5 and 6.6.

Corollary 6.2 The fuzzy topological sum space and the fuzzy topological quotient space of T_4-spaces are also T_4.

The following proposition shows that the finer fuzzy topological space of T_4 is also T_4.

Proposition 6.7 *Let (X, τ) be a T_4-space and let σ be a fuzzy topology on X finer than τ. Then (X, σ) is also T_4-space.*

Proof. Let F, G be disjoint closed subsets of X and let $\mathcal{N}_\tau(F)$, $\mathcal{N}_\sigma(F)$ be the fuzzy neighborhood filters of the spaces (X, τ), (X, σ), respectively, at F. From that (X, τ) is normal it follows $\mathcal{N}_\tau(F) \wedge \mathcal{N}_\tau(G)$ does not exist. Since σ is finer than τ, then $\mathcal{N}_\sigma(F) \leq \mathcal{N}_\tau(F)$ and $\mathcal{N}_\sigma(G) \leq \mathcal{N}_\tau(G)$ and thus $\mathcal{N}_\sigma(F) \wedge \mathcal{N}_\sigma(G) \leq \mathcal{N}_\tau(F) \wedge \mathcal{N}_\tau(G)$. Hence $\mathcal{N}_\sigma(F) \wedge \mathcal{N}_\sigma(G)$ does not exist and therefore (X, σ) is also normal. From Proposition 3.8 it follows that (X, σ) is T_1-space and thus it is T_4-space. □

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